Integrate sec x

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(A) $\int \sec x \, dx$

There are different solutions to integrate sec x. Readers are suggested to show that they are equivalent.

(1) $\int \sec x\, dx = \int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2}+x\right)} = \int \frac{dx}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \frac{1}{2}\int \frac{dx}{\frac{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}$

$= \int \frac{\frac{1}{2}\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)dx}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \int \frac{d\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \mathbf{\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c}$

(2) $\int \sec x\, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x dx}{\cos^2 x} = \int \frac{\cos x dx}{(1+\sin x)(1-\sin x)}$

Put $t = \sin x$, $\int \sec x\, dx = \int \frac{1}{(1+t)(1-t)} dt = \frac{1}{2}\int \left[\frac{1}{1+t} + \frac{1}{1-t}\right] dt = \frac{1}{2}[\ln|1+t| - \ln|1-t|] + c$

$= \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c = \mathbf{\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c}$

(3) $\int \sec x\, dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x} dx = \int \frac{(\sec x \tan x+ \sec^2 x)dx}{\sec x+\tan x} = \int \frac{d(\sec x+\tan x)}{\sec x+\tan x} = \mathbf{\ln|\sec x + \tan x| + c}$

(4) Put $\sec x = \cosh\theta$, $\tan x = \sqrt{\sec^2\theta - 1} = \sqrt{\cosh^2\theta - 1} = \sinh\theta$
Also, we can get $\sin x = \tanh\theta$ (please check yourselves)

Therefore $\sec^2 x\, dx = \cosh\theta\, d\theta \Longrightarrow dx = \frac{\cosh\theta d\theta}{\sec^2 x} = \frac{\cosh\theta d\theta}{\cosh^2\theta} = \frac{d\theta}{\cosh\theta}$

$\int \sec x\, dx = \int \cosh\theta \frac{d\theta}{\cosh\theta} = \int d\theta = \theta + c = \cosh^{-1}(\sec x) + c$

$\therefore \int \sec x\, dx = \mathbf{\cosh^{-1}(\sec x) + c = \sinh^{-1}(\tan x) + c = \tanh^{-1}(\sin x) + c}$

(5) Put $s = \sec x$, $ds = \sec x \tan x\, dx$, $\sec x\, dx = \frac{ds}{\tan x} = \frac{ds}{\sqrt{s^2-1}}$

$\int \sec x\, dx = \int \frac{ds}{\sqrt{s^2-1}} = \ln\left|s + \sqrt{s^2-1}\right| + c = \mathbf{\ln|\sec x + \tan x| + c}$

Proof of: $\int \frac{ds}{\sqrt{s^2-1}} = \ln\left|s + \sqrt{s^2-1}\right| + c$

Put $u^2 = s^2 - 1$, $2udu = 2sds \Longrightarrow \frac{du}{s} = \frac{ds}{u} = \frac{du + ds}{s+u} = \frac{d(s+u)}{s+u}$

$\int \frac{ds}{\sqrt{s^2-1}} = \int \frac{ds}{u} = \int \frac{d(s+u)}{s+u} = \ln|s+u| + c = \ln\left|s + \sqrt{s^2-1}\right| + c$

(6) Put $t=\tan\frac{x}{2}, dx=\frac{2dt}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}, \sec x=\frac{1+t^2}{1-t^2}$

$$\int \sec x\,dx=\int\frac{1+t^2}{1-t^2}\left[\frac{2dt}{1+t^2}\right]=\int\frac{2dt}{1-t^2}=\int\left[\frac{1}{1+t}+\frac{1}{1-t}\right]dt=\ln\left|\frac{1+t}{1-t}\right|+c=\mathbf{ln}\left|\frac{\mathbf{1+tan\frac{x}{2}}}{\mathbf{1-tan\frac{x}{2}}}\right|+\mathbf{c}$$

Also, putting $\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$, we have

$$\int \sec x\,dx=\mathbf{ln}\left|\frac{\mathbf{cos\frac{x}{2}+sin\frac{x}{2}}}{\mathbf{cos\frac{x}{2}-sin\frac{x}{2}}}\right|+\mathbf{c}$$

(B) $\int \mathbf{sec^n\,x\,dx}$ for different natural numbers n.

(1) $\int \sec^2 x\,dx=\tan x+c$

(2) $I=\int \sec^3 x\,dx=\int \sec x\sec^2 x\,dx=\int \sec x\,d(\tan x)$

$=\sec x\tan x-\int \tan x d(\sec x)$ (integation by parts)

$=\sec x\tan x-\int \tan x\sec x\tan x\,dx$

$=\sec x\tan x-\int \tan^2 x\sec x\,dx=\sec x\tan x-\int(\sec^2 x-1)\sec x\,dx$

$=\sec x\tan x-\int \sec^3 x\,dx+\int \sec x\,dx$

$=\sec x\tan x-I+\ln|\sec x+\tan x|$

$2I=\sec x\tan x+\ln|\sec x+\tan x|$

$$I=\int \sec^3 x\,dx=\frac{\mathbf{1}}{\mathbf{2}}[\mathbf{sec\,x tan\,x}+\mathbf{ln}|\mathbf{sec\,x}+\mathbf{tan\,x}|]+\mathbf{c}$$

(3) $\int \sec^4 x\,dx=\int(1+\tan^2 x)\sec^2 x\,dx=\int(1+\tan^2 x)d(\tan x)=\mathbf{tan\,x}+\frac{\mathbf{tan^3\,x}}{\mathbf{3}}+\mathbf{c}$

(4) Reduction formula for $\int \mathbf{sec^n\,x\,dx}$ (used mostly for n is odd)

$I_n=\int \sec^n x\,dx=\int \sec^{n-2} x\,d(\tan x)=\sec^{n-2} x\tan x-\int \tan x\,d(\sec^{n-2} x)$

$=\sec^{n-2} x\tan x-\int \tan x\,(n-2)\sec^{n-3} x\,(\sec x\tan x)dx$

$=\sec^{n-2} x\tan x-(n-2)\int \tan^2 x\sec^{n-2} x\,dx$

$=\sec^{n-2} x\tan x-(n-2)\int(\sec^2 x-1)\sec^{n-2} x\,dx$

$=\sec^{n-2} x\tan x-(n-2)I_n+(n-2)I_{n-2}$

$$\mathbf{I_n}=\frac{\mathbf{1}}{\mathbf{n-1}}[\mathbf{sec^{n-2}\,x\,tan\,x}+(\mathbf{n-2})\mathbf{I_{n-2}}]$$

(5) $I_5=\int \sec^5 x\,dx=\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}I_3=\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}\left[\frac{1}{2}(\sec x\tan x+I_1)\right]$

$$=\frac{\mathbf{1}}{\mathbf{4}}\mathbf{sec^3\,x\,tan\,x}+\frac{\mathbf{3}}{\mathbf{8}}[\mathbf{sec\,x\,tan\,x}+\mathbf{ln}|\mathbf{sec\,x}+\mathbf{tan\,x}|]+\mathbf{c}$$

(6) $\int \mathbf{sec^n\, x\, dx}$ **(n = 2m is even)**

$$\int \sec^{2m} x\, dx = \int \sec^{2m-2} x\, d(\tan x) = \int (1+\tan^2 x)^{m-1} d(\tan x)$$

$$= \int (1+u^2)^{m-1} du \quad (\text{where } u = \tan x)$$

$$= \int \sum_{r=0}^{m-1} C_r^{m-1} u^{2r}\, du = \sum_{r=0}^{m-1} C_r^{m-1} \int u^{2r} du = \sum_{r=0}^{m-1} C_r^{m-1} \frac{u^{2r+1}}{2r+1} + c$$

$$\int \mathbf{sec^{2m}\, x\, dx} = \sum_{\mathbf{r=0}}^{\mathbf{m-1}} \mathbf{C_r^{m-1}} \frac{\mathbf{(\tan x)^{2r+1}}}{\mathbf{2r+1}} + \mathbf{c}$$

(7) $\int \sec^6 x\, dx = \sum_{r=0}^{2} C_r^2 \frac{(\tan x)^{2r+1}}{2r+1} = C_0^2 \tan x + C_1^2 \frac{(\tan x)^3}{3} + C_2^2 \frac{(\tan x)^5}{5} + c$

$$= \mathbf{\tan x} + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{(\tan x)^3} + \frac{\mathbf{2}}{\mathbf{5}}\mathbf{(\tan x)^5} + \mathbf{c}$$

(C) $\int \sqrt{\mathbf{sec\, x}}\,\mathbf{dx}$ **(for interest)**

It is integrable, but cannot be expressed in terms of elementary functions. It involves elliptic integral of the first kind.

$\int \sqrt{\mathbf{sec\, x}}\,\mathbf{dx} = \mathbf{2\, F\left(\frac{x}{2}, 2\right)}$, where $F(x, m)$ is the elliptic integral of the first kind with parameter $m = k^2$.

Maclaurin expansion of $\sqrt{\sec x} \approx 1 + \frac{x^2}{4} + \frac{7x^4}{96} + \frac{139x^6}{5760} + \cdots$

$$\int \sqrt{\sec x}\, dx \approx \int \left(1 + \frac{x^2}{4} + \frac{7x^4}{96} + \frac{139x^6}{5760}\right) dx \approx \mathbf{x} + \frac{\mathbf{x^3}}{\mathbf{12}} + \frac{\mathbf{7x^5}}{\mathbf{480}} + \frac{\mathbf{139x^7}}{\mathbf{40320}} + \mathbf{c}$$